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Entanglement, Bose operators, coherent states and classical dynamical systems

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Abstract

Classical dynamical systems can be embedded into a Hilbert space description by using Bose operators and Glauber coherent states. Thus the embedding in a Hilbert space allows us to study entanglement of the states. We apply it here to first integrals which are expressed as states.

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Ordinary and partial differential equations and ordinary and partial difference equations can be embedded into linear equations in a Hilbert space using Bose operators and Bose field operators, respectively [1–4]. Here we consider systems of ordinary differential equations, autonomous systems of (nonlinear) difference equations and their embedding using Bose operators and Glauber coherent states. One of the basic tasks in the study of nonlinear dynamical systems is to find out whether or not the system is integrable. For systems of ordinary differential equations one has to find the first integrals (if any exists) and for difference equations one has to find invariants (if any exists). These questions can be investigated with the help of Bose operators and Glauber coherent states. Glauber coherent states [5] are defined by applying the displacement operator $D(\beta, \beta^*) := \exp(\beta b^\dagger - \beta^* b)$ to the vacuum state $|0\rangle$ with the straightforward extension to more than one Bose operator. We study whether first integrals and invariants both expressed as states in the Hilbert space are entangled [6–9].

There are other embedding techniques so that we can study the entanglement of classical systems. In the Koopman linearization we embed the system into a Hilbert space of square integrable functions [10]. Suzuki [11] showed that d-dimensional quantum systems can be embedded into a $(d + 1)$-dimensional classical system. Such an example is studied by Vedral [12]. Gottesman [13] studies the Bose–Hubbard model as a classical Boson model which is a Markov process. These techniques will not be considered here.

Consider the initial-value problem for the systems of first-order ordinary differential equations

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{V}(\mathbf{u}), \qquad \mathbf{u}(0) = \mathbf{u}_0. \tag{1}$$

It is assumed that $\mathbf{V} : \mathbf{R}^n \to \mathbf{R}^n$ is analytic. Let us now summarize the embedding [1–4]. We recall that the Bose operators satisfy the commutation relation

$$\left[b_j, b_k^\dagger\right] = \delta_{jk} I \qquad [b_j, b_k] = \left[b_j^\dagger, b_k^\dagger\right] = 0,$$

where I is the identity operator and $j, k = 1, 2, \ldots, n$, $b_k|\mathbf{0}\rangle = 0$ and

$$|\mathbf{0}\rangle \equiv |0\rangle \otimes \cdots \otimes |0\rangle$$

denotes the vacuum state. A complete set of state vectors may be written as a product of state vectors for each mode (Fock states)

$$|n_1\rangle \otimes |n_2\rangle \otimes \cdots \otimes |n_N\rangle,$$

where $n_j = 0, 1, \ldots, \infty$. We have

$$b_j^\dagger |n_1\rangle \otimes \cdots \otimes |n_j\rangle \otimes \cdots \otimes |n_N\rangle = \sqrt{n_j + 1}|n_1\rangle \otimes \cdots \otimes |n_j + 1\rangle \otimes \cdots \otimes |n_N\rangle$$
$$b_j |n_1\rangle \otimes \cdots \otimes |n_j\rangle \otimes \cdots \otimes |n_N\rangle = \sqrt{n_j}|n_1\rangle \otimes \cdots \otimes |n_j - 1\rangle \otimes \cdots \otimes |n_N\rangle.$$

We define the operator

$$M := \mathbf{b}^\dagger \cdot \mathbf{V}(\mathbf{b}) \equiv \sum_{j=1}^{n} b_j^\dagger V_j(\mathbf{b}). \tag{2}$$

Let

$$|\mathbf{u}(t)\rangle := \exp\left(-\tfrac{1}{2}|\mathbf{u}(t)|^2\right) \exp(\mathbf{u}(t) \cdot \mathbf{b}^\dagger)|\mathbf{0}\rangle \tag{3}$$

be a Glauber coherent state ($\mathbf{b}|\mathbf{u}(t)\rangle = \mathbf{u}(t)|\mathbf{u}(t)\rangle$), where

$$\mathbf{u}(t) \cdot \mathbf{b}^\dagger := \sum_{j=1}^{n} u_j(t) b_j^\dagger$$

and $\mathbf{u}$ satisfies equation (1). Furthermore $\langle \mathbf{0}|\mathbf{0}\rangle = 1$. If we define

$$|\widetilde{\mathbf{u}}(t)\rangle := \exp\left(\tfrac{1}{2}(|\mathbf{u}(t)|^2 - |\mathbf{u}_0|^2)\right)|\mathbf{u}(t)\rangle, \tag{4}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}|\widetilde{\mathbf{u}}(t)\rangle = M|\widetilde{\mathbf{u}}(t)\rangle \tag{5}$$

is the corresponding infinite system. The formal solution of (1) is given by

$$\mathbf{u}(\mathbf{u}_0, t) = \mathbf{u}_0 + \sum_{j=1}^{\infty} \frac{(-t)^j}{j!} \langle \mathbf{u}_0|[M, \ldots, [M, \mathbf{b}] \ldots]|\mathbf{u}_0\rangle. \tag{6}$$

To study the entanglement for a pure bipartite state $|\psi\rangle$ (i.e. $n = 2$) we study the von Neumann entropy of the reduced density operator of its one component [6–9], i.e.

$$E(|\psi\rangle\langle\psi|) = -\mathrm{tr}(\rho_1 \log_2(\rho_1)) = -\mathrm{tr}(\rho_2 \log_2(\rho_2)). \tag{7}$$

This means we first calculate the density operator $\rho_{12} = |\psi\rangle\langle\psi|$ from the pure bipartite state and then the partial trace to find ρ_1 and ρ_2. Properties such as concavity, subadditivity, strong subadditivity and the triangle inequality also follow for this measure of entanglement as they do for the finite-dimensional case provided the relevant quantities converge when we consider the infinite-dimensional system. To calculate the partial trace we can either use the number basis $|k\rangle$ with $k = 0, 1, 2, \ldots, \infty$ or coherent states $|\beta\rangle$. To test quantum nonlocality of the states, the displaced parity operator

$$\hat{\Pi}_{12}(\alpha, \beta) := D_1(\alpha) D_2(\beta)(-1)^{\hat{n}_1+\hat{n}_2} D_1^\dagger(\alpha) D_2^\dagger(\beta)$$

based on joint parity measurement can be used, where $\hat{n}_1 := b_1^\dagger b_1$ and $\hat{n}_2 := b_2^\dagger b_2$. Here $D_1(\alpha)$ and $D_2(\beta)$ are the unitary phase-space coherent displacement operators. Other entanglement measures such as entanglement of formation, distillable entanglement, relative entropy of entanglement and logarithmic negativity are difficult to implement for the continuous case [14].

The most studied case is the two-mode state

$$|\psi\rangle = \mathrm{e}^{r(b_1^\dagger b_2^\dagger - b_1 b_2)}|00\rangle,$$

where r is the squeezing parameter. This state can also be written as

$$|\psi\rangle = \frac{1}{\cosh(r)} \sum_{n=0}^{\infty} (\tanh(r))^n |n\rangle \otimes |n\rangle.$$

This equation is just the Schmidt decomposition of the state $|\psi\rangle$. Then the density operator is given by $\rho_{12} = |\psi\rangle\langle\psi|$ and after calculating the partial trace we find for the entanglement [15, 16]

$$E(r) = \cosh^2(r)\log_2(\cosh^2(r)) - \sinh^2(r)\log_2(\sinh^2(r)),$$

where $E(r = 0) = 0$. The entanglement of this state can be viewed as an entanglement between quadrature phases in the two modes (EPR entanglement) or as an entanglement between number and phase in two modes.

If we investigate the case with $n \geqslant 2$ we can also discuss whether the states given by the first integral are entangled or not. This is a question discussed in quantum computing [6–9]. We consider now the case $n = 2$ and first integrals of the dynamical system (1) which are represented by states in the Hilbert space. We apply the entropy of entanglement (sometimes just called the entanglement) given by (7) as a measure for entanglement. Let us first consider the Lotka–Volterra model

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = -u_1 + u_1 u_2 \qquad \frac{\mathrm{d}u_2}{\mathrm{d}t} = u_2 - u_1 u_2.$$

Here the associated vector field is given by

$$V = (-u_1 + u_1 u_2)\frac{\partial}{\partial u_1} + (u_2 - u_1 u_2)\frac{\partial}{\partial u_2}.$$

The corresponding Bose operator is given by

$$M^\dagger = \left(-b_1^\dagger + b_1^\dagger b_2^\dagger\right) b_1 + \left(b_2^\dagger - b_1^\dagger b_2^\dagger\right) b_2.$$

The first integral of the Lotka–Volterra model takes the form

$$H(\mathbf{u}) = u_1 u_2 \,\mathrm{e}^{-(u_1+u_2)} \equiv u_1 \,\mathrm{e}^{-u_1} u_2 \,\mathrm{e}^{-u_2}.$$

In the formulation with Bose operators the first integral of the Lotka–Volterra model is the state vector

$$b_1^\dagger b_2^\dagger \,\mathrm{e}^{-(b_1^\dagger + b_2^\dagger)}|00\rangle.$$

Consequently,

$$\left[\left(-b_1^\dagger + b_1^\dagger b_2^\dagger\right) b_1 + \left(b_2^\dagger - b_1^\dagger b_2^\dagger\right) b_2\right] b_1^\dagger b_2^\dagger \,\mathrm{e}^{-(b_1^\dagger + b_2^\dagger)}|00\rangle = 0$$

since $L_V H = 0$, where $L_V(\cdot)$ denotes the Lie derivative. The state given by the first integral for the Lotka–Volterra model is not entangled, i.e. it can be written as the product state

$$b_1^\dagger b_2^\dagger \,\mathrm{e}^{-(b_1^\dagger + b_2^\dagger)}|00\rangle = (b^\dagger \otimes I)(I \otimes b^\dagger)\,\mathrm{e}^{-(b^\dagger \otimes I + I \otimes b^\dagger)}|00\rangle = (b^\dagger\,\mathrm{e}^{-b^\dagger}) \otimes (b^\dagger\,\mathrm{e}^{-b^\dagger})(|0\rangle \otimes |0\rangle)$$

since $b_1 = b \otimes I$, $b_2 = I \otimes b$ and $\mathrm{e}^{-b^\dagger \otimes I - I \otimes b^\dagger} = \mathrm{e}^{-b^\dagger} \otimes \mathrm{e}^{-b^\dagger}$, where I denotes the identity operator. Thus calculating the entropy of entanglement defined by (7) provides $E = 0$.

We recall that if a pure bipartite state $|\psi\rangle$ can be written as $|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$, then $E = 0$.

An example of a system with an entangled first integral is the anharmonic oscillator

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = u_2, \qquad \frac{\mathrm{d}u_2}{\mathrm{d}t} = u_1 + u_1^3$$

with the first integral $u_1^2/2 + u_1^4/4 - u_2^2/2$ and the corresponding state (not yet normalized)

$$|\psi\rangle = \left(\tfrac{1}{2}\left(b_1^\dagger\right)^2 + \tfrac{1}{4}\left(b_1^\dagger\right)^4 - \tfrac{1}{2}\left(b_2^\dagger\right)^2\right)|00\rangle.$$

For this state we find that $E > 0$. Thus it is entangled using the definition given above.

Consider now the case $n > 2$. In this case the question arises for a measure of entanglement. For finite-dimensional systems for tripartite states and four qubit states the geometric invariant theory is utilized using the Lie group $SL(2, \mathbf{C})$. However this cannot be applied to the infinite-dimensional case. Thus we will apply the basic definition that the state is entangled if it cannot be written as any combinations of tensor products of states.

As an example, we consider an explicitly time-dependent first integral for the autonomous systems of first-order ordinary differential equations (1). We extend (1) to

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\lambda} = V(\mathbf{u}), \qquad \frac{\mathrm{d}t}{\mathrm{d}\lambda} = 1.$$

Then

$$W = \sum_{j=1}^{n} V_j(\mathbf{u}) \frac{\partial}{\partial u_j} + \frac{\partial}{\partial t}$$

is the corresponding vector field of the system. An explicitly time-dependent smooth function $H(\mathbf{u}(t), t)$ is a first integral of the system if $L_W H = 0$. The corresponding Bose operator of the vector field W is

$$W \mapsto M^\dagger := \sum_{j=1}^{n} V_j(\mathbf{b}^\dagger) b_j + b_{n+1},$$

where we have identified t with u_{n+1}. As an application, we consider an extended Lorenz model

$$\frac{\mathrm{d}u_1}{\mathrm{d}\lambda} = \sigma(u_3 - u_1) + \sigma s u_5 \qquad \frac{\mathrm{d}u_2}{\mathrm{d}\lambda} = -b u_2 + u_1 u_3$$
$$\frac{\mathrm{d}u_3}{\mathrm{d}\lambda} = -u_1 u_2 + r u_1 - b u_3 \qquad \frac{\mathrm{d}u_4}{\mathrm{d}\lambda} = u_1 u_5 - b\tau u_4 - b\tau u_2$$
$$\frac{\mathrm{d}u_5}{\mathrm{d}\lambda} = -u_1 u_4 + r u_1 - \tau u_5 + \tau u_3 \qquad \frac{\mathrm{d}t}{\mathrm{d}\lambda} = 1,$$

where $\sigma, r, b, s, \tau \in \mathbf{R}^+$, $t = u_6$ and $t(\lambda = 0) = 0$. Thus

$$M^\dagger = \sigma\left(b_3^\dagger - b_1^\dagger + s b_5^\dagger\right)b_1 + \left(-b b_2^\dagger - b_1^\dagger b_3^\dagger\right)b_2 + \left(-b_1^\dagger b_2^\dagger + r b_1^\dagger - b b_3^\dagger\right)b_3$$
$$+ \left(b_1^\dagger b_5^\dagger - b\tau b_4^\dagger - b\tau b_2^\dagger\right)b_4 + \left(-b_1^\dagger b_4^\dagger + r b_1^\dagger - \tau b_5^\dagger + \tau b_3^\dagger\right)b_5 + b_6.$$

For example for $\sigma = 1$, $b = 1$ and $\tau = 2$ we find the explicitly time-dependent first integral

$$H(\mathbf{u}(t), t) = \left(2sru_4 + 4sru_2 - ru_1^2 - (2s-1)\left(u_2^2 + u_3^2\right)\right)\exp(2t).$$

Consequently, we find that the first integral expressed in Bose operators assumes the form of the state vector

$$\left(2srb_4^\dagger + 4srb_2^\dagger - rb_1^{\dagger 2} - (2s-1)\left(b_2^{\dagger 2} + b_3^{\dagger 2}\right)\right)\exp\left(2b_6^\dagger\right)|\mathbf{0}\rangle.$$

Therefore

$$M^\dagger\left(2srb_4^\dagger + 4srb_2^\dagger - rb_1^{\dagger 2} - (2s-1)\left(b_2^{\dagger 2} + b_3^{\dagger 2}\right)\mathrm{e}^{2b_6^\dagger}|\mathbf{0}\rangle = 0.$$

Note the operator $b_5^\dagger$ does not appear in the state since u_5 is not present in the first integral. To check for entanglement, i.e. if the state cannot be written as a product state across the Hilbert space we write, using $b_1 = b \otimes I \otimes I \otimes I \otimes I \otimes I$ etc, the state in the tensor product form

$$\begin{aligned}(c_1(I \otimes I \otimes I \otimes b^\dagger) + c_2(I \otimes b^\dagger \otimes I \otimes I) + c_3(b^{\dagger 2} \otimes I \otimes I \otimes I)\\ + c_4(I \otimes b^{\dagger 2} \otimes I \otimes I) + c_5(I \otimes I \otimes b^{\dagger 2} \otimes I)) \otimes I \otimes \mathrm{e}^{2b^\dagger}|\mathbf{0}\rangle,\end{aligned}$$

where $c_1 = 2sr$, $c_2 = 4sr$, $c_3 = -r$, $c_4 = (2s-1)$, $c_5 = (2s-1)$ and we used that

$$\mathrm{e}^{2b_6^\dagger} = I \otimes I \otimes I \otimes I \otimes I \otimes \mathrm{e}^{2b^\dagger}.$$

Thus the state is a product state with respect to the fifth and sixth modes. Thus with respect to the fifth and sixth modes the state is not entangled. However for the first four modes the state is entangled. This can be seen when we consider the underlying polynomial

$$p(u_1, u_2, u_3, u_4) = 2sru_4 + 4sru_2 - ru_1^2 - (2s-1)\left(u_2^2 + u_3^2\right),$$

which cannot be written as any product of polynomials. For example the polynomial p cannot be written as $u_4 f(u_1, u_2, u_3)$ where f again is a polynomial, etc. No factorization is possible.

We could also start from entangled states as the first integral and then construct the dynamical system which admits this first integral.

We can also study the entanglement for the state $|\tilde{\mathbf{u}}(t)\rangle$ as the solution of (5) which corresponds to the solution of (1).

We can also introduce a time-dependent density operator $\rho(t)$ for the classical system (1). We define a density matrix as

$$\rho := \int_{\mathbf{C}^n} \mathrm{d}\mu(\mathbf{w})\, \mathrm{e}^{-|\mathbf{w}-\mathbf{u}_0|^2}|\mathbf{w}\rangle\langle\mathbf{w}|,$$

where $|\mathbf{w}\rangle$ is a coherent state and

$$\mathrm{d}\mu(\mathbf{w}) := \prod_{j=1}^{n} \frac{1}{\pi}\, d(\mathrm{Re}\, w_j)(d\, \mathrm{Im}\, w_j).$$

Thus the density matrix is Hermitian $\rho^\dagger = \rho$ and positive $\rho \geqslant 0$. Furthermore $\mathrm{tr}\,\rho = 1$ and

$$\langle\mathbf{b}(t)\rangle := \mathrm{tr}(\rho\mathbf{b}(t)) = \mathbf{u}(t),$$

where $\mathbf{b}(t)$ are the time-dependent Bose annihilation operators and $\mathbf{u}(t)$ satisfies the system of autonomous differential equations (1). Next we introduce the time-dependent density matrix

$$\rho(t) = \int_{\mathbf{C}^n} \mathrm{d}\mu(\mathbf{w})\, \mathrm{e}^{-|\mathbf{w}-\mathbf{u}_0|^2}\, \mathrm{e}^{tM}|\mathbf{w}\rangle\langle\mathbf{w}|\, \mathrm{e}^{tM^\dagger},$$

where M is the operator given by (2). Thus we can define a time-dependent entropy

$$S(t) := -\mathrm{tr}(\rho(t)\ln\rho(t)).$$

Entanglement can also be studied for the autonomous system of first-order ordinary difference equations

$$x_{1,t+1} = f_1(x_{1,t}, x_{2,t}), \qquad x_{2,t+1} = f_2(x_{1,t}, x_{2,t}),$$

where $t = 0, 1, 2, \ldots$ and we assume that f_1 and f_2 are analytic functions and $x_{1,0}, x_{2,0}$ are the initial values with $x_{1,t}, x_{2,t} \in \mathbf{R}$. They also can be embedded in a Hilbert space using Bose operators and coherent states [4]. The extension to higher dimensions is straightforward. Now we describe how invariants can be expressed as Bose operators. To embed this system into a Hilbert space using Bose operators $b_j^\dagger, b_j$ with $j = 1, 2$ we consider the Hilbert space states

$$|x_1, x_2, t\rangle := \exp\left(\tfrac{1}{2}\left(x_{1,t}^2 + x_{2,t}^2 - x_{1,0}^2 - x_{2,0}^2\right)\right)|x_{1,t}, x_{2,t}\rangle,$$

where $|x_{1,t}, x_{2,t}\rangle$ is the normalized coherent state

$$|x_{1,t}, x_{2,t}\rangle := \exp\left(-\tfrac{1}{2}\left(x_{1,t}^2 + x_{2,t}^2\right)\right)\exp\left(x_{1,t}b_1^\dagger + x_{2,t}b_2^\dagger\right)|\mathbf{0}\rangle.$$

Next we introduce the evolution operator

$$\hat{M} := \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{b_1^{\dagger j}}{j!}\frac{b_2^{\dagger k}}{k!}(f_1(b_1, b_2) - b_1)^j(f_2(b_1, b_2) - b_2)^k.$$

It follows that

$$|x_1, x_2, t + 1\rangle = \hat{M}|x_1, x_2, t\rangle,$$

where $t = 0, 1, 2, \ldots$. Thus the system of difference equations is mapped into a linear difference equation in a Hilbert space. The price to be paid for linearity is that we have to deal with Bose operators which are linear unbounded operators. Furthermore we have the eigenvalue equations

$$b_1|x_1, x_2, t\rangle = x_{1,t}|x_1, x_2, t\rangle$$

$$b_2|x_1, x_2, t\rangle = x_{2,t}|x_1, x_2, t\rangle$$

for the states given above, since $|x_{1,t}, x_{2,t}\rangle$ is a coherent state. Let $K(x_1, x_2)$ be an analytic function of x_1, x_2. Let $\hat{K}(b_1, b_2)$ be the corresponding operator. Thus we have

$$\hat{K}(b_1, b_2)|x_1, x_2, t\rangle = K(x_{1,t}, x_{2,t})|x_1, x_2, t\rangle.$$

It follows that

$$[\hat{K}, \hat{M}]|x_1, x_2, t\rangle = (K(x_{1,t+1}, x_{2,t+1}) - K(x_{1,t}, x_{2,t}))|x_1, x_2, t + 1\rangle,$$

where $[\hat{K}, \hat{M}] = \hat{K}\hat{M} - \hat{M}\hat{K}$. Thus $\hat{K}$ is an invariant, i.e.

$$K(x_{1,t+1}, x_{2,t+1}) = K(x_{1,t}, x_{2,t}),$$

if $[\hat{K}, \hat{M}] = 0$.

As an example consider the logistic equation

$$x_{t+1} = 2x_t^2 - 1, \qquad t = 0, 1, 2, \ldots$$

and $x_0 \in [-1, 1]$. All quantities of interest in chaotic dynamics can be calculated exactly. The logistic equation is an invariant of a class of second-order difference equations

$$x_{t+2} = g(x_t, x_{t+1}), \qquad t = 0, 1, 2, \ldots.$$

This means that if the logistic map is satisfied for a pair (x_t, x_{t+1}), then $x_{t+2} = g(x_t, x_{t+1})$ implies that (x_{t+1}, x_{t+2}) also satisfies the logistic map. In general, let

$$x_{t+1} = f(x_t), \qquad t = 0, 1, 2, \ldots$$

be a first-order difference equation. Then this equation is called an invariant of $x_{t+2} = g(x_t, x_{t+1})$, if

$$g(x, f(x)) = f(f(x)).$$

We find that the logistic map is an invariant of the trace map

$$x_{t+2} = 1 + 4x_t^2(x_{t+1} - 1).$$

The trace map plays an important role for the study of tight-binding Schrödinger equations with disorder. This second-order difference equation can be written as a first-order system of difference equations ($x_{1,t} \equiv x_t, x_{2,t} \equiv x_{t+1}$)

$$x_{1,t+1} = x_{2,t}, \qquad x_{2,t+1} = g(x_{1,t}, x_{2,t}).$$

After embedding the two maps into the linear unbounded operators $\hat{M}$ and $\hat{K}$ we can show that $[\hat{M}, \hat{K}] = 0$ using the commutation relation given above.

Another example is the Fibonacci trace map

$$x_{t+3} = 2x_{t+2}x_{t+1} - x_t.$$

This map admits the invariant

$$I(x_t, x_{t+1}, x_{t+2}) = x_t^2 + x_{t+1}^2 + x_{t+2}^2 - 2x_t x_{t+1} x_{t+2} - 1,$$

i.e. $I(x_t, x_{t+1}, x_{t+2}) = I(x_{t+1}, x_{t+2}, x_{t+3})$ for $t = 0, 1, 2, \ldots$. The Fibonacci trace map can be written as a system of three first-order difference equations. After embedding the two maps into the linear unbounded operators $\hat{M}$ and $\hat{K}$ for I we find that $[\hat{M}, \hat{K}] = 0$, since I is an invariant.

To summarize, we have shown that linear and nonlinear classical systems can be embedded into a Hilbert space using Bose operators and Glauber coherent states. The states (for example the states given by the first integrals) in this Hilbert space can be studied with respect to entanglement using the reduced density operator and the von Neumann entropy. We can extend the technique to (nonlinear) partial differential and difference equations [2, 3]. Here the conservation laws appear as states in a Hilbert space and thus entanglement can be studied. The extension of Glauber coherent states to generalized coherent states the so-called Klauder–Perelomov and Gazeau–Klauder coherent states [17] would also be worthwhile to study.

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